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Gravitational Radiation from an Isolated System of N Bodies in Higher Multipole Moments

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Explicit expressions are given of the energy, angular and linear momentum flux (higher multipole moments) in the linear approximation from an isolated system of N bodies, within general relativity.

1. INTRODUCTION

The exact Einstein field equations of an N-body system can be written

$$\frac{\partial^2}{\partial x^l \partial x^m}[(-g)(g^{ij}g^{lm} - g^{il}g^{jm})] = 16\pi[(-g)(T^{ij} + t^{ij})] \tag{1.1}$$

where $\Theta^{ij} = (-g)(T^{ij} + t^{ij})$ is the Landau–Lifshitz complex and

$$T^{ij}(\mathbf{x}, t) = \sum_{\nu=1}^{n} \frac{m_\nu}{(-g)^{1/2}} \frac{dx_\nu{}^i}{ds} \frac{dx_\nu{}^j}{dt} \delta(\mathbf{x} - \mathbf{x}_\nu(t)) \tag{1.2}$$

is, in curvilinear coordinates, the energy-momentum tensor of the N point masses, g is the determinant of the metric tensor g_{ij}, $\mathbf{x}_\nu$ are the source points, $\mathbf{x}$ the field point, and t^{ij} is the pseudo-energy momentum tensor.

Latin indices (space-time) take the values 0, 1, 2, 3, while Greek indices (space) take the values 1, 2, 3. Also, we set $G = c = 1$.

We assume that the N-bodies system is changing slowly, the field is everywhere weak, and we do not come closely up to any source point.

Using the Newtonian terms of the Landau–Lifshitz complex

$$\Theta^{ij} = (-g)(T^{ij} + t^{ij}) \tag{1.3}$$

we can write

$$\Theta_N^{ij} = (T^{ij} + t^{ij})_N \tag{1.4}$$

where, when this is done, the pseudotensor t^{ij} is defined by

$$t_N^{\alpha\beta} = \frac{1}{16\pi}\left[4\frac{\partial V}{\partial x^\alpha}\frac{\partial V}{\partial x^\beta} - 2\delta_{\alpha\beta}\left(\frac{\partial V}{\partial x^\gamma}\right)^2\right] \tag{1.5}$$

or (Dionysiou, 1974)

$$t_N^{\alpha\beta} = -\frac{1}{2}\sum_{m'}\sum_{m''}\frac{m'm''(x_\alpha - x'_\alpha)(x_\beta - x'_\beta)}{|\mathbf{x} - \mathbf{x}'|^3}\,\delta(\mathbf{x} - \mathbf{x}'') \qquad \text{(mod. div.)} \tag{1.6}$$

$$t_N^{00} = -\frac{7}{8\pi}\left(\frac{\partial V}{\partial x^\alpha}\right)^2, \qquad V = \sum_{m'}\frac{m'}{|\mathbf{x} - \mathbf{x}'|} \tag{1.7}$$

and

$$t_N^{0\alpha} = 0 \tag{1.8}$$

It is certainly true that the dominant terms of Θ_N^{ij} are tensors and explicitly defined as (Dionysiou, 1974)

$$\Theta_N^{00} = \sum_{m'} m'\delta(\mathbf{x} - \mathbf{x}') \tag{1.9}$$

$$\Theta_N^{0\alpha} = \sum_{m'} m'u'_\alpha\delta(\mathbf{x} - \mathbf{x}') \tag{1.10}$$

$$\Theta_N^{\alpha\beta} = \sum_{m'} m'u'_\alpha u'_\beta\delta(\mathbf{x} - \mathbf{x}') + t^{\alpha\beta} \tag{1.11}$$

or

$$\Theta_N^{\alpha\beta} = \sum_{m''} m''u''_\alpha u''_\beta\delta(\mathbf{x} - \mathbf{x}'') - \frac{1}{2}\sum_{m'}\sum_{m''}\frac{m'm''(x_\alpha - x'_\alpha)(x_\beta - x'_\beta)}{|\mathbf{x} - \mathbf{x}'|^3}\,\delta(\mathbf{x} - \mathbf{x}'') \qquad \text{(mod. div.)} \tag{1.12}$$

Further we define

$$\Theta'^{\alpha\beta}_N = \int_{\text{all space}} \Theta_N^{\alpha\beta} d\mathbf{x} \tag{1.13}$$

2. DERIVATION OF THE METRIC FAR FROM THE N-BODIES SYSTEM

We start with the definition (Infeld and Plebanski, 1960)

$$(-g)^{1/2}g^{ij} = n^{ij} + \gamma^{ij} \qquad (g^{ij} = n^{ij} + h^{ij},\ |h^{ij}| \ll 1) \tag{2.1}$$

where η^{ij} is the metric tensor, which can be written as a matrix

$$(\eta^{ij}) = (\eta_{ij}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

We impose on γ^{ij} the gauge condition

$$\gamma^{ij}_{,j} = 0 \tag{2.2}$$

where the comma denotes partial differentiation.

Hence, the field equations (1.1) in the weak-field region far outside the source reduce to (Fock, 1957)

$$\eta^{lm}\gamma^{ij}_{,lm} = 16\pi\Theta^{ij}_N \tag{2.3}$$

Thus, Einstein's equations are equivalent to

$$\gamma^{ij}(\mathbf{x}, t) = 4 \int_{\text{all space}} \frac{\Theta^{ij}_N[\mathbf{x}', t - |\mathbf{x} - \mathbf{x}'|]}{|\mathbf{x} - \mathbf{x}'|} d\mathbf{x}' \tag{2.4}$$

where

$$|\mathbf{x} - \mathbf{x}'| = \left[\sum_{\alpha=1}^{3} (x_\alpha - x'_\alpha)\right]^{1/2}, \qquad d\mathbf{x}' = dx'_1 dx'_2 dx'_3$$

The integral (2.4) is valid for any field point $(\mathbf{x}, t)$ even in the region between the particles. We interpret equations (2.4) as the gravitational radiation produced by the source Θ^{ij}_N. The occurrence in equations (2.4) of the time argument $t - |\mathbf{x} - \mathbf{x}'|$ shows that gravitational effects propagate with unit velocity, that is to say, with the speed of light. Also, Θ^{ij} unlike T^{ij} need not necessarily vanish outside the source region (Section 4).

Locating the origin of coordinates inside the region between the particles of the source system, we have for field points far from it ($|\mathbf{x}| \gg |\mathbf{x}'|$) (Papapetrou, 1971)

$$\gamma^{ij}(r, t) = \frac{4}{r} \int_{\text{all space}} \Theta^{ij}_N[\mathbf{x}', t - r + (\mathbf{n}\cdot\mathbf{x}')]d\mathbf{x}' + O(r^{-2}) \tag{2.5}$$

where $|\mathbf{x} - \mathbf{x}'| = r - \mathbf{n}\cdot\mathbf{x}' + O(r^{-1})$, $|\mathbf{x} - \mathbf{x}'|^{-1} = 1|r + (\mathbf{x}\cdot\mathbf{x}')/|\mathbf{x}|^3 + \cdots$, $r = |\mathbf{x}|$, $\mathbf{n} = \mathbf{x}|r$ is the unit vector in the direction of propagation.

Now, we expand the integrand of equation (2.5) in powers of $\mathbf{n}\cdot\mathbf{x}'$, i.e.,

$$\Theta^{ij}_N[\mathbf{x}', t - r + (\mathbf{n}\cdot\mathbf{x}')] = \Theta^{ij}_N[\mathbf{x}', (t - r)] + (\mathbf{n}\cdot\mathbf{x}')\frac{\partial}{\partial t}\Theta^{ij}_N[\mathbf{x}', (t - t)]$$

$$+ \frac{(\mathbf{n}\cdot\mathbf{x}')^2}{2!}\frac{\partial^2}{\partial t^2}\Theta^{ij}_N[\mathbf{x}', (t - r)] + \cdots \tag{2.5a}$$

then equation (2.5) can be written as an expansion:

$$\gamma^{ij}(r, t) = \frac{4}{r}\sum_{\nu=0}^{\infty}\frac{1}{\nu!}\int_{\text{all space}}\left\{\frac{\partial^\nu}{\partial t^\nu}\Theta^{ij}_N[\mathbf{x}', (t - r)]\right\}(\mathbf{n}\cdot\mathbf{x}')^\nu d\mathbf{x}' + O(r^{-2}) \tag{2.6}$$

If the motion of the particles is sufficiently slow, i.e., there is a limitation on the speed of the particles ($u' \ll 1$, $c = 1$), on their acceleration and on higher time derivatives, then equation (2.6) may be replaced by an expansion as

$$\gamma^{ij}(r, t) = \frac{4}{r} \sum_{\nu=0}^{\infty} \frac{1}{\nu!} \frac{\partial^\nu}{\partial t^\nu} \int_{\text{all space}} \Theta_N^{ij}[\mathbf{x}', (t - r)](\mathbf{n}\cdot\mathbf{x}')^\nu d\mathbf{x}' + O(t^{-2}) \quad (2.7)$$

The convergence of integrals (2.4), (2.6), and (2.7) is given by Section 4. Henceforth, the sources Θ_N^{ij} have to be considered at a retarded time $t - r$.

3. N-POINT MASSES IN SLOW MOTION RADIATING GRAVITATIONAL WAVES

3.1. Energy. For slow motion systems, the only significant contributions to equation (2.4) come from a region of size $L \sim R \ll \lambda\!\!\!^{-}$, where R is the size of the source ($R = |\mathbf{x}'|$ max) and $\lambda\!\!\!^{-}$ ($\lambda\!\!\!^{-} = \lambda/2\pi$) the reduced wavelength. We confine our attention to field points $\mathbf{x}$ far outside the source region, i.e.,

$$|\mathbf{x}| = r \gg L \geqslant |\mathbf{x}'| \qquad (R \ll \lambda\!\!\!^{-} \ll |\mathbf{x}| = r)$$

We use equation (2.7) and the obvious identity

$$\mathbf{n}\cdot\mathbf{x}' = n_{\kappa_1}x'_{\kappa_1} = n_{\kappa_2}x'_{\kappa_2} = \cdots = n_{\kappa_\nu}x'_{\kappa_\nu}$$

and can then write

$$\begin{aligned}\gamma^{ij}(r, t) = {} & \frac{4}{r}\int \Theta_N^{ij} d\mathbf{x}' + \frac{4}{r} n_{\kappa_1} \frac{\partial}{\partial t}\int \Theta_N^{ij} x'_{\kappa_1} d\mathbf{x}' + \frac{4}{r} n_{\kappa_1} n_{\kappa_2} \frac{1}{2!}\frac{\partial^2}{\partial t^2} \\ & \times \int \Theta_N^{ij} x'_{\kappa_1} x'_{\kappa_2} d\mathbf{x}' + \cdots + \frac{4}{r} n_{\kappa_1} n_{\kappa_2} n_{\kappa_3} \cdots n_{\kappa_\nu} \frac{1}{\nu!}\frac{\partial^\nu}{\partial t^\nu} \\ & \times \int \Theta_N^{ij} x'_{\kappa_1} x'_{\kappa_2} x'_{\kappa_3} \cdots x'_{\kappa_\nu} d\mathbf{x}' + O(r^{-2}) \end{aligned} \quad (3.1)$$

where the terms on the right are 1-pole, 2-pole, 4-pole, and so on (Papapetrou, 1971).

One can put the conservation laws with the help of the special form

$$\Theta_{N,j}^{ij} = 0 \quad (3.2)$$

Applying equations (3.2) one obtains the identities

$$\frac{\partial^2 \Theta_N^{00}}{\partial t^2} \equiv \frac{\partial^2 \Theta_N^{\alpha\beta}}{\partial x'_\alpha \partial x'_\beta}$$

$$\frac{\partial^2}{\partial t^2}(\Theta_N^{00} x'_\alpha x'_\beta) \equiv \frac{\partial^2}{\partial x'_\gamma \partial x'_\delta}(\Theta_N^{\gamma\delta} x'_\alpha x'_\beta) - 2\frac{\partial}{\partial x'_\gamma}(\Theta_N^{\gamma\alpha} x'_\beta + \Theta_N^{\gamma\beta} x'_\alpha) + 2\Theta_N^{\alpha\beta} \quad (3.3)$$

(Misner et al., 1973). We suppose Θ_N^{ij} is spatially confined as it is required by equation (2.4); then the first and second terms on the right-hand side of equations (3.3) are seen to be zero when we integrate over all space by an application of Gauss' theorem. (Gaussian flux integrals are valid only in asymptotically flat regions of space-time and in asymptotically Minkowskian coordinates.) Hence the results are (see Section 4)

$$\int \Theta_N^{\alpha\beta}(\mathbf{x}', t - r)d\mathbf{x}' = \frac{1}{2}\frac{\partial^2}{\partial t^2}\int \Theta_N^{00}(\mathbf{x}', t - r)x'_\alpha x'_\beta d\mathbf{x}' \tag{3.4}$$

and

$$\int \Theta_N^{\alpha\beta}x'_\kappa d\mathbf{x}' = \frac{1}{6}\frac{\partial^2}{\partial t^2}\int \Theta_N^{00}x'_\alpha x'_\beta x'_\kappa d\mathbf{x}'$$
$$-\frac{1}{3}\frac{\partial}{\partial t}\int [(\Theta_N^{0\kappa}x'_\beta - \Theta_N^{00}x'_\kappa)x'_\alpha + (\Theta_N^{0\kappa}x'_\alpha - \Theta_N^{0\alpha}x'_\kappa)x'_\beta]d\mathbf{x}' \tag{3.5}$$

since from equations (3.2), we have

$$\int (\Theta_N^{\alpha\beta}x'_\kappa + \Theta_N^{\beta\kappa}x'_\alpha + \Theta_N^{\alpha\kappa}x'_\beta)d\mathbf{x}' = \frac{1}{2}\frac{\partial^2}{\partial t^2}\int \Theta_N^{00}x'_\alpha x'_\beta x'_\kappa d\mathbf{x}' \tag{3.6}$$

and

$$\int (\Theta_N^{\alpha\kappa}x'_\beta - \Theta_N^{\alpha\beta}x'_\kappa)d\mathbf{x}' = \frac{\partial}{\partial t}\int (\Theta_N^{0\kappa}x'_\beta - \Theta_N^{0\beta}x'_\kappa)x'_\alpha d\mathbf{x}' \tag{3.7}$$

Using the transverse–traceless (TT) gauge condition (Misner, Thorne, and Wheeler, 1973), i.e.,

$$h^{i0} = 0, \qquad h^{\alpha\beta}_{,\beta} = 0, \qquad h^{\alpha\alpha} = 0 \tag{3.8}$$

then the gravitational radiation is completely described by the (gauge-invariant) transverse–traceless part of the metric perturbation h^{ij}; hence since

$$h^{ij} = \gamma^{ij} - \tfrac{1}{2}\eta^{ij}\gamma + \text{nonlinear expressions} \tag{3.9}$$

we have from equations (3.8) and (3.9)

$$h^{ij}_{TT} = \gamma^{ij}_{TT} \tag{3.9a}$$

The effective stress-energy tensor for the outgoing waves has the form

$$\Theta^{00}_{GW} = \frac{1}{32\pi}\langle\gamma^{\alpha\beta}_{TT,0}\gamma^{\alpha\beta}_{TT,0}\rangle \tag{3.10}$$

where $\langle\ \rangle$ denotes an average over several wavelengths (in accord with one's inability to localize the gravitational radiation inside a wavelength) and $\gamma^{\alpha\beta}_{TT}$ means the gauge invariant transverse-traceless part of γ^{ij}. Equation (3.10) is the formula for radiation from a nearly Newtonian slow-motion source (Misner et al., 1973).

Then the total power crossing a large sphere of radius r at time t is

$$E_{GW}(r, t) = \int \Theta_{GW}^{00} r^2 dO \tag{3.11}$$

where $ds = r^2 dO = r^2 \sin\theta d\theta d\varphi$, $0 \leqslant \theta \leqslant \pi$, $0 \leqslant \varphi \leqslant 2\pi$.

Hence

$$E_{GW}(r, t) = \frac{1}{32\pi} \int \langle \gamma_{TT,0}^{\alpha\beta} \gamma_{TT,0}^{\alpha\beta} \rangle r^2 dO \tag{3.12}$$

Now, using equations (3.1), (3.4), and (3.8) we take

$$\begin{aligned} \gamma_{TT}^{\alpha\beta}(r, t) = \frac{4}{r} \Bigg(& \frac{1}{2} \frac{\partial^2}{\partial t^2} \int \Theta_N^{00} x'_\alpha x'_\beta d\mathbf{x}' + n_{\kappa_1} \Bigg\{ \frac{1}{6} \frac{\partial^3}{\partial t^3} \int \Theta_N^{00} x'_\alpha x'_\beta x'_{\kappa_1} d\mathbf{x}' \\ & - \frac{1}{3} \frac{\partial^2}{\partial t^2} \int [(\Theta_N^{0\kappa_1} x'_\beta - \Theta_N^{0\beta} x'_{\kappa_1}) x'_\alpha + (\Theta_N^{0\kappa_1} x'_\alpha - \Theta_N^{0\alpha} x'_\kappa) x'_\beta] d\mathbf{x}' \Bigg\} \\ & + \frac{1}{2} n_{\kappa_1} \eta_{\kappa_2} \frac{\partial^2}{\partial t^2} \int \Theta_N^{\alpha\beta} x'_{\kappa_1} x'_{\kappa_2} d\mathbf{x}' + \cdots \\ & + \frac{1}{\nu!} n_{\kappa_1} n_{\kappa_2} n_{\kappa_3} \cdots n_{\kappa\nu} \frac{\partial^\nu}{\partial t^\nu} \int \Theta_N^{\alpha\beta} x'_{\kappa_1} x'_{\kappa_2} x'_{\kappa_3} \cdots x'_{\kappa_\nu} d\mathbf{x}' \Bigg) + O(r^{-2}) \end{aligned} \tag{3.13}$$

The second term of equation (3.13), after integrating, becomes

$$\frac{2}{r} \eta_{\kappa_1} \frac{\partial^2}{\partial t^2} \left(\sum_m m u_\alpha x_\beta x_{\kappa_1} + \sum_m m x_\alpha u_\beta x_{\kappa_1} - \sum_m m x_\alpha x_\beta u_\kappa \right) \tag{3.14}$$

Considering equation (3.13) we readily verify that

$$\gamma_{TT}^{\alpha\beta}(r, t) = \frac{2}{r} \sum_{\nu=0}^{\infty} n_{\kappa_1} n_{\kappa_2} n_{\kappa_3} \cdots n_{\kappa_\nu} \frac{\partial^2}{\partial t^2} I^{\alpha\beta\kappa_1\kappa_2\kappa_3\cdots\kappa_\nu}(t - r) + O(r^{-2}) \tag{3.15}$$

where we have put

$$I^{\alpha\beta}(t) = \sum_m m x_\alpha x_\beta \tag{3.16}$$

$$I^{\alpha\beta\kappa_1}(t) = \sum_m m u_\alpha x_\beta x_{\kappa_1} + \sum_m m x_\alpha u_\beta x_{\kappa_1} - \sum_m m x_\alpha x_\beta u_{\kappa_1} \tag{3.17}$$

and

$$I^{\alpha\beta\kappa_1\kappa_2\cdots\kappa_\nu}(t) = \frac{2}{\nu!} \frac{d^{\nu-2}}{dt^{\nu-2}} \int \Theta_N^{\alpha\beta} x'_{\kappa_1} x'_{\kappa_2} x'_{\kappa_3} \cdots x'_{\kappa_\nu} d\mathbf{x}', \qquad \nu \geqslant 2 \tag{3.18}$$

or

$$I^{\alpha\beta\kappa_1\kappa_2\cdots\kappa_\nu}(t) = \frac{2}{\nu!} \frac{d^{\nu-2}}{dt^{\nu-2}} (\Theta_N'^{\alpha\beta} x_{\kappa_1} x_{\kappa_2} \cdots x_{\kappa_\nu}), \qquad \nu \geqslant 2 \tag{3.19}$$

using equation (1.13).

If we define

$$D^{\alpha\beta\kappa_1\kappa_2\cdots\kappa_\nu} \equiv I^{\alpha\beta\kappa_1\kappa_2\cdots\kappa_\nu} - \tfrac{1}{3}\delta_{\alpha\beta} I^{\gamma\gamma\kappa_2\kappa_2\cdots\kappa_\nu} \tag{3.20}$$

which gives

$$D^{\alpha\alpha\kappa_1\kappa_2\cdots\kappa_\nu} = 0 \tag{3.21}$$

$$D_{TT}^{\alpha\beta\kappa_1\kappa_2\cdots\kappa_\nu} = I_{TT}^{\alpha\beta\kappa_1\kappa_2\cdots\kappa_\nu} \tag{3.22}$$

then equation (3.15) becomes

$$\gamma_{TT,0}^{\alpha\beta} = \frac{2}{r} \sum_{\nu=0}^{\infty} n_{\kappa_1} n_{\kappa_2} n_{\kappa_3} \cdots n_{\kappa_\nu} \ddot{D}_{TT}^{\alpha\beta\kappa_1\kappa_2\cdots\kappa_\nu} + O(r^{-2}) \tag{3.23}$$

By making use of equations (3.10) and (3.23) we find that

$$\begin{aligned}\Theta_{GW}^{00} = \frac{1}{8\pi r^2} \Big\langle \sum_{\nu,\rho=0}^{\infty} & n_{\kappa_1} n_{\kappa_2} \cdots n_{\kappa_\nu} n_{\lambda_1} n_{\lambda_2} \cdots n_{\lambda_\rho} \\ & \times (\dddot{D}^{\alpha\beta\kappa_1\kappa_2\cdots\kappa_\nu} \dddot{D}^{\alpha\beta\lambda_1\lambda_2\ldots\lambda_\nu} - 2n_\gamma n_\delta \dddot{D}^{\beta\gamma\kappa_1\kappa_2\cdots\kappa_\nu} \\ & \times \dddot{D}^{\beta\delta\lambda_1\lambda_2\ldots\lambda_\rho} + \tfrac{1}{2} n_\alpha n_\beta n_\gamma n_\delta \dddot{D}^{\alpha\beta\kappa_1\kappa_2\cdots\kappa_\rho} \dddot{D}^{\gamma\delta\lambda_1\lambda_2\ldots\lambda_\rho}) \Big\rangle \end{aligned} \tag{3.24}$$

since the simplest one is

$$\dddot{D}^{\alpha\beta}\dddot{D}^{\alpha\beta} - 2n_\gamma n_\delta \dddot{D}^{\beta\gamma}\dddot{D}^{\beta\delta} + \tfrac{1}{2} n_\alpha n_\beta n_\gamma n_\delta \dddot{D}^{\alpha\beta}\dddot{D}^{\gamma\delta}$$

Now, putting equation (3.24) into equation (3.12), we get the final answer for the total power, i.e.,

$$\begin{aligned} E_{GW}(r, t) = \langle \tfrac{1}{5} \dddot{D}^{\alpha\beta}\dddot{D}^{\alpha\beta} + \tfrac{1}{105} & \\ \times (11 \dddot{D}^{\alpha\beta\kappa}\dddot{D}^{\alpha\beta\kappa} & - 6\dddot{D}^{\alpha\beta\beta}\dddot{D}^{\alpha\kappa\kappa} - 6\dddot{D}^{\alpha\beta\kappa}\dddot{D}^{\alpha\kappa\beta} \\ & + 22\dddot{D}^{\alpha\beta}\dddot{D}^{\alpha\beta\kappa\kappa} - 24\dddot{D}^{\alpha\beta}\dddot{D}^{\alpha\kappa\kappa\beta})\rangle \\ + (\text{terms with } 8, 10, 12, \ldots \text{ indices}) & \end{aligned} \tag{3.25}$$

To evaluate $E_{GW}(r, t)$, we use the results

$$\int n_{\kappa_1} n_{\kappa_2} n_{\kappa_3} \cdots n_{\kappa_\nu} dO = \frac{4\pi}{1, 3, 5, \ldots, (2\nu + 1)} \Delta\kappa_1\kappa_2\kappa_3 \cdots \kappa_\nu \qquad (\nu \text{ even}) \tag{3.26}$$

where $\Delta\kappa_1\kappa_2\kappa_3 \cdots \kappa_\nu$ means all distinct permutation of $\delta_{\kappa_1\kappa_2}$. Also,

$$\int n_{\kappa_1} n_{\kappa_2} n_{\kappa_3} \cdots n_{\kappa_\nu} dO = 0 \qquad (\nu \text{ odd}) \tag{3.27}$$

3.2. Angular Momentum. The density of angular momentum in quadrupole–quadrupole moments is given by Misner et al., (1973), i.e.,

$$\mathscr{F}^{\alpha} = \frac{1}{8\pi r^2} e^{\alpha\beta\gamma} \langle -6n_\beta \ddot{D}^{\gamma\epsilon} \dddot{D}^{\epsilon\lambda} n_\lambda + 9n_\beta \ddot{D}^{\gamma\epsilon} n_\epsilon n_\lambda \dddot{D}^{\lambda\kappa} n_\kappa \rangle \tag{3.28}$$

where $\langle\ \rangle$ denotes an average over several wavelengths.

The integral of this quantity over a large sphere of radius r, enclosing the system of N bodies, is the total angular momentum being transported outward per unit time, i.e.,

$$\mathscr{T}^{\alpha} = \int_{\text{sphere}} \mathscr{F}^{\alpha} r^2 dO \tag{3.29}$$

where $ds = r^2 dO = t^2 \sin\theta d\theta d\varphi$, $0 \leqslant \theta \leqslant \pi$, $0 \leqslant \varphi \leqslant 2\pi$.

Now, the generalization of equation (3.28), i.e., using higher multipole moments, gives that

$$\mathscr{F}^{\alpha} = \frac{1}{8\pi r^2} e^{\alpha\beta\gamma} \Big\langle \sum_{\mu,\rho=0}^{\infty} n_{\kappa_1} n_{\kappa_2} \cdots n_{\kappa_\mu} n_{\lambda_1} n_{\lambda_2} \cdots n_{\lambda_\rho} (-6 n_\beta n_\lambda \ddot{D}^{\gamma\epsilon\kappa_1\kappa_2\cdots\kappa_\mu} \dddot{D}^{\epsilon\lambda\lambda_1\lambda_2\cdots\lambda_\rho} + 9_{n_\beta n_\epsilon n_\lambda n_\kappa} \ddot{D}^{\gamma\epsilon\kappa_1\kappa_2\cdots\kappa_\mu} \dddot{D}^{\lambda\kappa\lambda_1\lambda_2\cdots\lambda_\rho}) \Big\rangle \tag{3.30}$$

Here, from equations (3.28) and (3.29), using the integrals (3.26) and (3.27) we get in quadrupole–quadrupole moments

$$\mathscr{T}^{\alpha} = -\tfrac{2}{5} e^{\alpha\beta\gamma} \langle \ddot{D}^{\gamma\epsilon} \dddot{D}^{\beta\epsilon} \rangle \tag{3.31}$$

In higher multipole moments, from equations (3.29) and (3.30) with the aid of the integrals (3.26) and (3.27), we get that

$$\begin{aligned} \mathscr{T}^{\beta\gamma} = \tfrac{2}{5} \{ & \langle \ddot{D}^{\beta\epsilon} \dddot{D}^{\gamma\epsilon} - \ddot{D}^{\gamma\epsilon} \dddot{D}^{\beta\epsilon} \rangle \\ & + \tfrac{1}{14} \langle [4(\ddot{D}^{\beta\epsilon\nu} \dddot{D}^{\gamma\epsilon\nu} - \ddot{D}^{\gamma\epsilon\nu} \dddot{D}^{\beta\epsilon\nu}) + 4\dddot{D}^{\epsilon\nu\nu}(\ddot{D}^{\beta\epsilon\gamma} - \ddot{D}^{\gamma\epsilon\beta}) \\ & + \ddot{D}^{\gamma\epsilon\nu}(3\dddot{D}^{\beta\nu\epsilon} - 4\dddot{D}^{\epsilon\nu\beta}) - \ddot{D}^{\beta\epsilon\nu}(3\dddot{D}^{\gamma\nu\epsilon} - 4\dddot{D}^{\epsilon\nu\gamma}) \\ & + 3(\ddot{D}^{\gamma\epsilon\epsilon} \dddot{D}^{\beta\nu\nu} - \ddot{D}^{\beta\epsilon\epsilon} \dddot{D}^{\gamma\nu\nu})] \rangle \} \\ & + (\text{terms with } 8, 10, 12, \ldots \text{ indices}) \end{aligned} \tag{3.32}$$

where

$$\mathscr{T} = (\mathscr{T}^{23}, \mathscr{T}^{31}, \mathscr{T}^{12})$$

Equation (3.32) gives the total angular momentum of the N-bodies system being transported outward per unit time. A similar, but less general, result has previously been derived by Papapetrou (1971).

3.3. Linear Momentum Flux. Considering the outflux of linear momentum, one may concentrate on the flux of the radial component η_α, i.e.,

$$P^{\alpha} = \int_{\text{sphere}} \Theta^{00}_{GW} r^2 n_\alpha dO \tag{3.33}$$

where P^{α} is the total outflux per unit time of the αth component of momentum. The integral being taken over a large sphere of radius r with n_{α} the three components of the outward normal and the differential solid angle.

Finally, from equations (3.24), (3.26), (3.27), and (3.33) we get that

$$P^{\alpha} = \tfrac{1}{105}\langle 22\dddot{D}^{\beta\gamma}\ddddot{D}^{\beta\gamma\alpha} - 12\dddot{D}^{\beta\gamma}\ddddot{D}^{\alpha\beta\gamma} - 12\dddot{D}^{\alpha\beta}\ddddot{D}^{\beta\gamma\gamma}\rangle$$
$$+ \text{(terms with 7, 9, 11, \ldots indices)} \qquad (3.34)$$

which is the same result in quadrupole–octupole moments as Dionysiou (1974, 1975, 1977a).

In conclusion, we mention that parallel results for an isolated perfect fluid have been obtained by the author (Dionysiou, 1977b).

4. TO JUSTIFY THE CONVERGENCE OF THE INTEGRALS

The retarded integrals equations have been based on the assumptions that the source Θ_N^{ij} is spatially confined [a remark of importance is that we have $t^{00} = t^{0\alpha} = 0$ from equations (1.9) and (1.10). Hence, out of the bound material system, there are contributions from $t^{\alpha\beta}$ only], since Θ_N^{ij} gives significant contributions to the integrals come only from a region of size $|\mathbf{x}'| \ll \lambda\!\!\!^{-}$. Now, in the near zone ($|\mathbf{x}'| \ll |\mathbf{x}| \ll \lambda\!\!\!^{-}$) but far from the source, $\Theta_N^{\alpha\beta} = t^{\alpha\beta}$ varies as $1/r^4$. In the radiation zone ($|\mathbf{x}| \gg \lambda\!\!\!^{-}$) $\Theta_N^{\alpha\beta} = t^{\alpha\beta}$ varies as $1/r^2$. The last integrals are ignored in our calculations as second-order effects because they have nothing to do with the emission process itself (Misner et al., 1973). We have to stress that it is valid since one can neglect the gravitational influence of the energy density of the waves as a second-order effect. Generally, we can put forth the following argument: Instead of an infinite wave train, we may consider a short pulse of gravitational waves. Then, one can verify without difficulty that the surface integrals at very large distance vanish (and the volume integrals vanish also), since the pulse has still not reached it. Of course, other general arguments can be presented about the convergence of these integrals (Ehlers et al., 1976).

APPENDIX

In order to check directly equation (3.4), we have from equations (1.9) and (3.16) that

$$\frac{1}{2}\frac{\partial^2}{\partial t^2}\int \Theta_N^{00}x'_{\alpha}x'_{\beta}d\mathbf{x}' = \frac{1}{2}\frac{d^2I_{\alpha\beta}}{dt^2} \qquad (A.1)$$

Also, from equations (1.12) and (1.13) we obtain

$$\int \Theta_N^{\alpha\beta} d\mathbf{x} = \int \left[\sum_{m''} m'' u''_\alpha u''_\beta \delta(\mathbf{x} - \mathbf{x}'') - \frac{1}{2} \sum_{m'} \sum_{m''} \frac{m'm''(x_\alpha - x'_\alpha)(x_\beta - x'_\beta)}{|\mathbf{x} - \mathbf{x}'|^3} \times \delta(\mathbf{x} - \mathbf{x}'') \right] d\mathbf{x}$$

$$= \sum_m m u_\alpha u_\beta - \frac{1}{2} \sum_{m'} \sum_{m''} \frac{m'm''(x''_\alpha - x'_\alpha)(x''_\beta - x'_\beta)}{|\mathbf{x}'' - \mathbf{x}'|^3}$$

$$= \sum_m m u_\alpha u_\beta - \frac{1}{2} \sum_{m} \sum_{m'} \frac{mm'(x_\alpha - x'_\alpha)(x_\beta - x'_\beta)}{|\mathbf{x} - \mathbf{x}'|^3} = 2Y_{\alpha\beta} + B_{\alpha\beta} \quad \text{(A.2)}$$

where $Y_{\alpha\beta}$ and $B_{\alpha\beta}$ denote the kinetic energy and the potential energy tensors, respectively. Then, from equations (A.1) and (A.2)

$$\frac{1}{2} \frac{d^2 I_{\alpha\beta}}{dt^2} = 2Y_{\alpha\beta} + B_{\alpha\beta} \quad \text{(A.3)}$$

which follows from the standard form of the tensor *Virial theorem.*

Here, we define [equation (1.6)]

$$f(\mathbf{x}, t) \equiv g(\mathbf{x}, t) \qquad \text{(modulo divergence)} \quad \text{(A.4)}$$

if the two functions f, g differ by the divergence of a vector, which vanishes sufficiently rapidly at infinity, so that their integrals over the whole space are equal (assuming that they exist).

REFERENCES

Dionysiou, D. D. (1974). *Int. J. Theor. Phys.*, **10**, 355.

Dionysiou, D. D. (1975). *Lett. Nuovo Cimento*, **13**, 86.

Dionysiou, D. D. (1977a). *J. Phys. A: Math. Nucl. Gen.*, **10**, 969.

Dionysiou, D. D. (1977b). *Astrophys. Space Sci.*, **52**, 17.

Ehlers, J., Rosenblum, A., Goldberg, J. N., and Havas, P. (1976). *Astrophys. J. Lett.* **208**, L77.

Fock, V. (1957). *Rev. Mod. Phys.*, **29**, 325.

Infeld, L., and Plebanski, J. (1960). *Motion and Relativity*, p. 78, Polska Acad. Nauk, Mongrafie, Fizyczue, Warsaw.

Misner, C. W., Thorne, K. S., and Wheeler, J. A. (1973). *Gravitation*, pp. 947–950, 989–1001, Freeman, San Francisco.

Papapetrou, A. (1971). *Ann. Inst. Henri Poincaré*, **14**, 79.